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APPROXIMATE CONSTRUCTION OF ATTAINABILITY SETS OF CONTROL SYSTEMS WITH INTEGRAL CONSTRAINTS ON THE CONTROLS†

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The approximate construction of attainability sets of control systems with quadratic integral constraints on the controls is considered. It is assumed that a control system is non-linear with respect to the phase variable and linear with respect to the variable which describes the controlling action. The approximation of the attainability sets of a control system is accomplished in several stages. The latter class of controls generates a finite number of trajectories of the system. The trajectories of the system are then replaced by Euler broken lines. An estimate of the accuracy of the Hausdorff distance between the attainability set and the set which has been approximately constructed is obtained. © 1999 Elsevier Science Ltd. All rights reserved.

Control systems with integral constraints [1–5] and geometric constraints [6–12] on the control resources have been considered previously.

1. Suppose a control system is given and its behaviour in the interval $I = [t_0, \theta]$ $(t_0 < \theta < \infty)$ is described by the equation

$$\dot{x}(t) = f(t, x(t)) + B(t, x(t)) \cdot u(t), \quad x(t_0) = x_0 \tag{1.1}$$

where $x \in R^n$ is the n-dimensional phase vector of the system, u is the r-dimensional control vector, $f(t, x)$ is an n-dimensional vector function and $B(t, x)$ is an $(n \times r)$-dimensional matrix function.

It is assumed that the realizations $u(t)$, $t \in I$ of the control u are limited by the constraints

$$\int_{t_0}^{\theta} \|u(t)\|^2 dt \leq \mu_0^2 \tag{1.2}$$

where $\|\cdot\|$ denotes a Euclidean norm and $\mu_0 \geq 0$ is the constraint on the control resources. It is also assumed that the following conditions are satisfied.

A. The functions $f(t, x)$ and $B(t, x)$ are continuous with respect to the totality of the variables t, x, and also for any bounded closed domain $D \subset I \times R^n$ Lipschits constants $L_i = L_i(D) \in (0, \infty)$ $(i = 1, 2)$ exist such that

$$\|f(t, x^*) - f(t, x_*)\| \leq L_1 \|x^* - x_*\|$$

$$\|B(t, x^*) - B(t, x_*)\| \leq L_2 \|x^* - x_*\|$$

for any (t, x^*), (t, x_*) from D.

B. Constants $\gamma_i \in (0, \infty)$ $(i = 1, 2)$ exist such that

$$\|f(t, x)\| \leq \gamma_1(1 + \|x\|), \quad \|B(t, x)\| \leq \gamma_2$$

for any $(t, x) \in I \times R^n$.

Here, $\|B\|$ is the Euclidean norm of matrix B. By a permissible control, $u(\cdot) = \{u(t), t \in I\}$, we mean any quadratically integrable function $u(\cdot)$ $(u(\cdot) \in L_2[t_0, \theta])$ which satisfies inequality (1.2). The class of all permissible controls $u(\cdot)$ is denoted by U.

By the solution of Eq. (1.1), which corresponds to a control $u(\cdot) \in U$, we mean the absolutely

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continuous vector function of $x(t)$, $t \in I$ which satisfies this equation almost everywhere in the interval I.

We shall call the solution of Eq. (1.1), which corresponds to a control $u(\cdot) \in U$, the motion of system (1.1) which is generated by the control $u(\cdot)$. The set of all motions of system (1.1) corresponding to all possible $u(\cdot) \in U$ is denoted by $X(t_0, x_0)$. We assume that

$$X(t; t_0, x_0) = \{x(t) \in R^n : x(\cdot) \in X(t_0, x_0)\}$$

$$Z(t_0, x_0) = \{(t, x(t)) \in I \times R^n : x(\cdot) \in X(t_0, x_0)\}$$

where $X(t; t_0, x_0)$ is called the attainability set of control system (1.1) with constraints (1.2) corresponding to an instant of time t and the set $Z(t_0, x_0)$ is called the integral funnel of system (1.1). It is obvious that the following equality is satisfied

$$Z(t_0, x_0) = \{(t, X(t; t_0, x_0)) : t \in I\}$$

where we put $(t, X) = \{(t, x) : x \in X\}$.

For the following arguments, we find the domain $D \subset I \times R^n$ in which the integral funnel $Z(t_0, x_0)$ is contained. For simplicity, we choose this domain to be cylindrical and put

$$D = \{(t, x) \in I \times R^n,\ \|x\| \leq r\},\ r = h(\theta)(1 + c\gamma_1(\theta - t_0))$$
$$h(t) = \|x_0\| + \gamma_1(t - t_0) + \gamma_2\mu_0\sqrt{t - t_0},\ c = \exp[\gamma_1(\theta - t_0)] \tag{1.3}$$

Assertion 1. The inclusion $Z(t_0, x_0) \subset D$ holds and the set D is defined by relation (1.3).

Proof. Suppose $x(\cdot)$ is an arbitrary motion of system (1.1), which is generated by a certain control $u(\cdot) \in U$. The representation

$$x(t) = x_0 + \int_{t_0}^{t} (f(\tau, x(\tau)) + B(\tau, x(\tau)) \cdot u(\tau))d\tau,\quad t \in I$$

holds.

By virtue of condition **B**, we have

$$\|x(t)\| \leq h(t) + \int^{t} \psi(\tau) \cdot \|x(\tau)\| d\tau,\quad \psi(t) = \gamma_1,\quad t \in I$$

The functions $x(t)$, $h(t)$, $\psi(t)$, $t \in I$ in this inequality satisfy the conditions of Gronwall's lemma [13, p. 219]. Using this lemma and taking account of the fact that the function $h(t)$ increases monotonically in I, we have

$$\|x(t)\| \leq h(\theta)(1 + c\int_{t_0}^{t} \gamma_1 d\tau) \leq h(\theta)(1 + c\gamma_1(\theta - t_0)),\quad t \in I \tag{1.4}$$

Since $x(\cdot)$ is an arbitrary motion of system (1.1), we obtain that the assertion holds from (1.4).

Everywhere in the following arguments, we shall have in mind the cylinder (1.3) as the set D.

2. Suppose $H \in (0, \infty)$. We now introduce into consideration the set U_H of all controls $u(\cdot) \in U$ for which

$$\|u(t)\| \leq H,\quad t \in I \tag{2.1}$$

The set of all $x \in R^n$, at which the motions of system (1.1) which are generated by all possible controls $u(\cdot) \in U_H$ arrive at the instant of time θ, is denoted by the symbol $X_H(\theta; t_0, x_0)$. We will find the upper limit of the Hausdorff distance between the set $X(\theta; t_0, x_0)$ and $X_H(\theta; t_0, x_0)$. We put

$$c_* = L_1(\theta - t_0) + L_2(\theta - t_0)^{1/2}\mu_0,\quad c_0 = 1 + c_* \exp[c_*] \tag{2.2}$$

Assertion 2. The Hausdorff distance $\alpha(X(\theta; t_0, x_0), X_H(\theta; t_0, x_0)) \to 0$ when $H \to \infty$ and, what is more, the following inequality holds

$$\alpha(X(t;t_0,x_0),X_H(t;t_0,x_0)) \leq 2\gamma_2 \frac{\mu_0^2}{H} c_0, \quad t \in I \tag{2.3}$$

Proof. Suppose the arbitrary controls $u(\cdot) \in U$ and $\tilde{u}(\cdot) \in U_H$ are chosen. They generate the motions $x(t)$ and $\tilde{x}(t)$ of system (1.1) in I respectively which satisfy the inequality

$$\begin{aligned} \|x(t)-\tilde{x}(t)\| \leq \int_{t_0}^{t} \|f(\tau,x(\tau))-f(\tau,\tilde{x}(\tau))\| d\tau + \int_{t_0}^{t} \|(B(\tau,x(\tau))-B(\tau,\tilde{x}(\tau)))u(\tau)\| d\tau + \\ + \int_{t_0}^{t} \|B(\tau,\tilde{x}(\tau))(u(\tau)-\tilde{u}(\tau))\| d\tau, \quad t \in I \end{aligned} \tag{2.4}$$

On taking account of condition A, we obtain from (2.4)

$$\varepsilon(t) \leq h(t) + \int_{t_0}^{t} \psi(\tau)\varepsilon(\tau)d\tau, \quad t \in I \tag{2.5}$$

$$\varepsilon(t) = \|x(t)-\tilde{x}(t)\|, \quad \psi(t) = L_1 + L_2\|u(t)\|$$

$$h(t) = \int_{t_0}^{t} \|B(\tau,\tilde{x}(\tau))\| \cdot \|u(\tau)-\tilde{u}(\tau)\| d\tau$$

We now make a certain correction with respect to the pair of controls $u(\cdot)$, $\tilde{u}(\cdot)$ and, in fact, we initially select a control $u(\cdot) \in U$, and, using this control, we form a control $\tilde{u}(\cdot) \in U_H$

$$\tilde{u}(t) = \begin{cases} u(t), & \text{if } \|u(t)\| \leq H \\ u(t)\|u(t)\|^{-1}H, & \text{if } \|u(t)\| > H \end{cases} \tag{2.6}$$

We assume that $\Omega_t = \{\tau \in [t_0, t] : \|u(\tau)\| > H\}$. Then, $[t_0, t]\backslash\Omega_t : \|u(\tau)\| \leq H\}$ and, according to (2.6), $\|u(\tau) - \tilde{u}(\tau)\| = 0$ when $\tau \in [t_0, t]\backslash\Omega_t$.

The inequality

$$\begin{aligned} h(t) = \int_{\Omega_t} \|B(\tau,\tilde{x}(\tau))\| \cdot \|u(\tau)-\tilde{u}(\tau)\| d\tau \leq \left(\int_{\Omega_t} \|B(\tau,\tilde{x}(\tau))\|^2 d\tau\right)^{1/2} \times \\ \times \left(\int_{\Omega_t} \|u(\tau)-\tilde{u}(\tau)\|^2 d\tau\right)^{1/2} \end{aligned} \tag{2.7}$$

holds.

We also have the inequality

$$H^2\mu(\Omega_t) \leq \int_{\Omega_t} \|u(\tau)\|^2 d\tau \leq \int_{\Omega_t} \|u(\tau)\|^2 d\tau + \int_{[t_0,t]\backslash\Omega_t} \|u(\tau)\|^2 d\tau \leq \mu_0^2$$

from which it follows that $\mu(\Omega_t) \leq \mu_0^2/H^2$. Here, $\mu(\Omega_t)$ is the Lebesgue measure of the set Ω_t. The inequality

$$\begin{aligned} h(t) \leq (\gamma_2^2\mu(\Omega_t))^{1/2}[(\int_{\Omega_t} \|u(\tau)\|^2 d\tau)^{1/2} + (\int_{\Omega_t} \|\tilde{u}(\tau)\|^2 d\tau)^{1/2})] \leq \\ \leq \gamma_2\mu(\Omega_t)^{1/2} 2(\mu_0^2)^{1/2} \leq 2\gamma_2\mu_0^2/H, \quad t \in I \end{aligned}$$

then follow from this and from (2.7).

On taking account of the inequality

$$\begin{aligned} \int_{t_0}^{t} \psi(\tau)d\tau = \int_{t_0}^{t} (L_1 + L_2\|u(\tau)\|)d\tau \leq L_1(t-t_0) + L_2(t-t_0)^{1/2}(\int_{t_0}^{t} \|u(\tau)\|^2 d\tau)^{1/2} \leq \\ \leq L_1(\theta-t_0) + L_2(\theta-t_0)^{1/2}\mu_0, \quad t \in I \end{aligned} \tag{2.8}$$

we obtain from inequality (2.5), using Gronwall's lemma

$$\varepsilon(t) \leqslant (1+c_* \exp[c_*]) \cdot 2\gamma_2 \frac{\mu_0^2}{H} \leqslant 2\gamma_2 \frac{\mu_0^2}{H} c_0, \quad t \in I \tag{2.9}$$

The constants $c_* > 0$, $c_0 > 0$ are defined by relations (2.2).

Thus, it has been shown that, for any control $u(\cdot) \in U$ which generates a motion $x(\cdot)$ of system (1.1), a control $\tilde{u}(\cdot) \in U_H$ is found which generates the motion $\tilde{x}(\cdot)$ which satisfies inequality (2.9). Since $U_H \subset U$, the fact that the assertion holds from this and from (2.9).

3. Now, taking estimate (2.3) into consideration, we can reduce the problem of the approximate calculation of the attainability set $X(t;\, t_0, x_0)$ to the problem of the approximate calculation of the attainability set $X_H(t;\, t_0, x_0)$, $t \in I$. The class U_H represents the set of controls which are limited by the composite constraints (1.2) and (2.1).

Bearing in mind the approximate calculation of the attainability set $X_H(t;\, t_0, x_0)$, we now try to narrow down the class of controls U_H. This is accomplished in three stages: we initially narrow down the class U_H to a certain class $\hat{U}_H$ of piecewise-constant controls, we then narrow down the class $\hat{U}_H$ to the class $\check{U}_H$ of piecewise-constant controls $u(\cdot)$, the norms $\|u(t)\|$ of the values $u(t)$ of which lie in a defined uniform mesh and, finally, we narrow down the class $\check{U}_H$ to the final class of controls $\tilde{U}_H$ for which not only the norms of the values of the controls but these values themselves in a local sense are uniformly arranged in a certain mesh. Note that each subsequent class of controls is more convenient in a certain sense for calculating the attainability sets which they generate. For instance, the final class $\tilde{U}_H$ is the most convenient for calculating the attainability sets. It generates a finite number of motions of system (1.1).

Each time, on passing from one class of controls to the following narrower class of controls, we shall show, using Gronwall's lemma, that this following class of controls approximates the preceding class of controls quite well, that is, the attainability sets corresponding to these classes are sufficiently close to one another.

We now introduce into the treatment a subdivision $\Gamma = \{t_0, t_1, \ldots, t_N = \theta\}$ of the interval $I = [t_0, \theta]$ such that henceforth

$$t_{i+1} - t_i = (\theta - t_0)/N = \Delta; \quad i = 0, 1, \ldots, N-1$$

We will now consider the class $\hat{U}_H$ of all possible piecewise-constant controls $\hat{u}(\cdot) \in U_H$, the intervals of constancy of which are the half intervals $\hat{I}_i = [t_i, t_{i+1})$ of the subdivision of Γ. We will denote the attainability set of system (1.1) which is generated by this class $\hat{U}_H$ and which corresponds to an instant of time $t \in I$ by $\hat{X}_H(t;\, t_0, x_0)$ and assume that

$$K = \max_{(t,x)\in D} \|f(t,x)\| \tag{3.1}$$

$$\varphi(\Delta) = (1+K)\Delta + \gamma_2 \mu_0 \Delta^{1/2} \tag{3.2}$$

$$\omega^*(\Delta) = \max_{|t^* - t_*| \leqslant \Delta,\, \|x^* - x_*\| \leqslant \Delta} \|B(t^*, x^*) - B(t_*, x_*)\| \tag{3.3}$$

$$\varkappa(\Delta) = 2\omega^*(\varphi(\Delta))(\theta - t_0)^{1/2}\mu_0 + 2\gamma_2\mu_0\Delta^{1/2}$$
$$(t^*, x^*) \in D, \quad (t_*, x_*) \in D, \quad \Delta \geqslant 0 \tag{3.4}$$

Assertion 3. The following inequality holds

$$\alpha(X_H(t; t_0, x_0),\ \hat{X}_H(t; t_0, x_0)) \leqslant c_0 \varkappa(\Delta), \quad t \in I \tag{3.5}$$

The constant $c_0 > 0$ is defined by relation (2.2).

Proof. Suppose $u(\cdot)$ is an arbitrary control from U_H. We set the control $\hat{u}(\cdot)$, which is specified by the equality

$$\hat{u}(t)=\frac{1}{\Delta}\int_{t_i}^{t_{i+1}} u(\tau)d\tau, \quad t\in \hat{I}_i \tag{3.6}$$

in correspondence to it.

It is obvious from the construction that

$$\|\hat{u}(t)\|\leq \frac{1}{\Delta}V_{i1}\leq H, \quad t\in\hat{I}_i, \quad V_{in}=\int_{t_i}^{t_{i+1}}\|u(\tau)\|^n d\tau, \quad n=1,\ 2$$

On taking account of the inequality $V_{i1}\leq \Delta^{1/2}V_{i2}^{1/2}$, we obtain that $\Delta\ ||\ \hat{u}(t)\ ||^2\leq V_{i2}$. By virtue of the constancy of the function $\hat{u}(\cdot)$ in the interval $\hat{I}_i$, the equality

$$\Delta\|\hat{u}(t)\|^2=\int_{t_i}^{t_{i+1}}\|\hat{u}(\tau)\|^2 d\tau, \quad t\in\hat{I}_i$$

is satisfied whereupon we obtain the inequality

$$\int_{t_0}^{\theta}\|\hat{u}(t)\|^2 d\tau\leq\int_{t_0}^{\theta}\|u(\tau)\|^2 d\tau\leq\mu_0^2$$

This means that $\hat{u}(\cdot)\in U_{\hat{H}}$.

Now, suppose $x(\cdot)$ and $\hat{x}(\cdot)$ are the motions of system (1.1) which are generated in I by the controls $u(\cdot)$ and $\hat{u}(\cdot)$, respectively. For these motions, the following equality is satisfied

$$\|x(t)-\hat{x}(t)\|=\int_{t_0}^{t}\|f(\tau,x(\tau))-f(\tau,\hat{x}(\tau))\|d\tau+\int_{t_0}^{t}\|(B(\tau,x(\tau))-$$
$$-B(\tau,\hat{x}(\tau)))u(\tau)\|d\tau+\left\|\int_{t_0}^{t}B(\tau,\hat{x}(\tau))(u(\tau)-\hat{u}(\tau))d\tau\right\|, \quad t\in I$$

By virtue of condition A, from this we obtain

$$\begin{aligned}&\varepsilon(t)\leq h(t)+\int_{t_0}^{t}\psi(\tau)\varepsilon(\tau)d\tau, \quad t\in I\\&\varepsilon(t)=\|x(t)-\hat{x}(t)\|, \quad \psi(\tau)=L_1+L_2\|u(\tau)\|\\&h(t)=\left\|\int_{t_0}^{t}B(\tau,\hat{x}(\tau))\cdot(u(\tau)-\hat{u}(\tau))d\tau\right\|\end{aligned} \tag{3.7}$$

We shall now obtain some estimates for the subsequent proof. The inequalities

$$\|\hat{x}(\tau)-\hat{x}(t_i)\|\leq\left\|\int_{t_i}^{\tau}(f(s,\hat{x}(s))+B(s,\hat{x}(s))\hat{u}(s))ds\right\|\leq\int_{t_i}^{\tau}\|f(s,\hat{x}(s))\|ds+\int_{t_i}^{\tau}\|B(s,\hat{x}(s))\|\cdot\|\hat{u}(s)\|ds\leq$$
$$\leq K(\tau-t_i)+\gamma_2(\tau-t_i)^{1/2}\left(\int_{t_i}^{\tau}\|\hat{u}(s)\|^2 ds\right)^{1/2}\leq K\Delta+\gamma_2\Delta^{1/2}\mu_0, \quad \tau\in\hat{I}_i$$

follow from condition B and notation (3.1).

By virtue of the notation of (3.2), from here we obtain

$$|\tau-t_i|\leq\varphi(\Delta), \quad \|\hat{x}(\tau)-\hat{x}(t_i)\|\leq\varphi(\Delta), \quad \tau\in\hat{I}_i \tag{3.8}$$

Hence, in the notation of (3.3), it follows that

$$\|B(\tau,\hat{x}(\tau))-B(t_i,\hat{x}(t_i))\|\leq\omega^*(\varphi(\Delta)), \quad \tau\in\hat{I}_i \tag{3.9}$$

Having obtained the required relations, we now estimate the magnitude of $h(t)$ from equality (3.7). For $t \in \hat{I}_k$, the following inequality is satisfied

$$h(t) \leq \left\| \int_{t_0}^{t_k} Y(\tau)d\tau \right\| + \left\| \int_{t_k}^{t} Y(\tau)d\tau \right\|, \quad t \in I, \quad Y(t) = B(t, \hat{x}(t)) \cdot (u(t)) - \hat{u}(t)) \tag{3.10}$$

By virtue of inequality (3.9), the estimate for the first integral

$$\begin{aligned} \left\| \int_{t_0}^{t_k} Y(\tau)d\tau \right\| &\leq \sum_{i=0}^{k-1} \left\| \int_{t_i}^{t_{i+1}} Y(\tau)d\tau \right\| = \sum_{i=0}^{k-1} \left\| \int_{t_i}^{t_{i+1}} (B(\tau, \hat{x}(\tau)) - B(t_i, \hat{x}(t_i)))(u(\tau) - \hat{u}(\tau))dt \right\| \leq \\ &\leq \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \omega^*(\varphi(\Delta))(\|u(\tau)\| + \|\hat{u}(\tau)\|)d\tau \leq \omega^*(\varphi(\Delta)) \cdot 2(\theta - t_0)^{1/2} \mu_0 \end{aligned} \tag{3.11}$$

is true.

For the second integral, we obtain

$$\left\| \int_{t_k}^{t} Y(\tau)d\tau \right\| \leq \int_{t_k}^{t} \gamma_2(\|u(\tau)\| - \|\hat{u}(\tau)\|)d\tau \leq 2\gamma_2 \Delta^{1/2} \mu_0, \quad t \in \hat{I}_k \tag{3.12}$$

Then, using relations (3.11) and (3.12), from inequality (3.10) we obtain the estimate

$$h(t) \leq 2\omega^*(\varphi(\Delta))(\theta - t_0)^{1/2} \mu_0 + 2\gamma_2 \mu_0 \Delta^{1/2}, \quad t \in I$$

On applying Gronwall's lemma to inequality (3.7), we obtain

$$\varepsilon(t) \leq (1 + c_* \exp[c_*])\varkappa(\Delta) \leq c_0 \varkappa(\Delta), \tag{3.13}$$

The constants $c_* > 0$, $c_0 > 0$ are defined by relation (2.2), and $\varkappa(\Delta) > 0$ is defined by relation (3.4). Since $\hat{U}_H \subset U_H$, the validity of the assertion follows from relation (3.13).

4. We now introduce the subdivision $\Gamma^* = \{y_0 = 0, y_1, \ldots, y_R = H^2\}$ of the interval $[0, H^2]$ such that

$$y_{j+1} - y_j = H^2 / R = \Delta^*; \quad j = 0, 1, \ldots, R-1$$

The class of all possible controls $\check{u}(\cdot) \in \hat{U}_H$, for which the values $y(t) = \|\check{u}(t)\|^2$, $t \in I$ are contained in the set Γ^*, is denoted by $\check{U}_H$. Hence, the condition $\|\check{u}(t)\| = \text{const}$, $t \in \hat{I}_i$ is satisfied for any control $\check{u}(\cdot) \in \check{U}_H$. What is more, $\|\check{u}(t)\|^2 = y_{j_i}$, where y_{j_i} is a certain value from Γ^*. Here and henceforth $i = 0, 1, \ldots, N-1$.

The attainability set of system (1.1) which is generated by this class and which corresponds to the instant of $t \in I$ is denoted by $\check{X}_H(t; t_0, x_0)$.

Assertion 4. The following inequality holds

$$\alpha(\hat{X}_H(t; t_0, x_0), \check{X}_H(t; t_0, x_0)) \leq c_0 \gamma_2 \sqrt{\Delta^*}(\theta - t_0), \quad t \in I \tag{4.1}$$

The constant $c_0 > 0$ is defined by relation (2.2).

Proof. Suppose $\hat{u}(\cdot)$ is an arbitrary control from $\hat{U}_H$. Then, $\hat{u}(t) = \hat{u}_i = \text{const}$ when $t \in \hat{I}_i$.
We shall put

$$\check{u}(t) = \sqrt{y_{j_i}} \|\hat{u}_i\|^{-1} \hat{u}_i, \quad t \in \hat{I}_i$$

where y_{j_i} are the points of the subdivision Γ^* such that $\|\hat{u}_i\|^2 \in [y_{j_i}, y_{j_{i+1}})$. It is obvious that $\|\check{u}(t)\|^2 = y_{j_i} \leq \|\hat{u}_i\|^2 = \|\hat{u}(t)\|^2$ for all $t \in \hat{I}_i$. Consequently

$$\int_{t_0}^{\theta}\left\|\check{u}(t)\right\|^2 dt \leq \int_{t_0}^{\theta}\|\hat{u}(t)\|^2 dt \leq \mu_0^2$$

Hence we have proved that $\check{u}(\cdot) \in \check{U}_H$.
The following relation is satisfied

$$\|\hat{u}(t)-\check{u}(t)\| = \|\hat{u}_i\| - \sqrt{y_{j_i}} \leq \sqrt{y_{j_i+1}} - \sqrt{y_{j_i}} \leq \sqrt{\Delta^*} \quad t \in \hat{I}_i \tag{4.2}$$

Now, suppose $\hat{x}(t)$ and $\check{x}(t)$, $t \in I$ are two motions of system (1.1) which are generated by the controls $\hat{u}(t)$, respectively. The equality

$$\hat{x}(t)-\check{x}(t) = \int_{t_0}^{t}(f(\tau,\hat{x}(\tau)) - f(\tau,\check{x}(\tau)))d\tau + \int_{t_0}^{t}(B(\tau,\hat{x}(\tau)) -$$

$$-B(\tau,\check{x}(\tau)))\hat{u}(\tau)d\tau + \int_{t_0}^{t}B(\tau,\check{x}(\tau))(\hat{u}(\tau)-\check{u}(\tau))d\tau, \quad t \in I$$

is true or, according to condition A

$$\varepsilon(t) \leq h(t) + \int_{t_0}^{t}\psi(\tau)\varepsilon(\tau)d\tau, \quad t \in I \tag{4.3}$$

$$\varepsilon(t) = \left\|\hat{x}(t-\check{x}(t)\right\|, \quad \psi(\tau) = L_1 + L_2\|\hat{u}(\tau)\|$$

$$h(t) = \int_{t_0}^{t}\left\|B(\tau,\check{x}(\tau))\right\| \cdot \left\|\hat{u}(\tau)-\check{u}(\tau)\right\|d\tau$$

The estimate

$$h(t) \leq \gamma_2\sqrt{\Delta^*}(\theta - t_0), \quad t \in I$$

is true for the function $h(t)$ by virtue of condition **B** and inequality (4.2).
Then, on applying Gronwall's lemma to inequality (4.3), we obtain

$$\varepsilon(t) \leq c_0\gamma_2\sqrt{\Delta^*}(\theta - t_0), \quad t \in I \tag{4.4}$$

The constant $c_0 > 0$ is defined by relation (2.2).
Since the inclusion $\check{U}_H \subset U_H$ is true, the validity of the assertion follows from relation (4.4).

5. Suppose $S = \{u \in R^r: \|u\| = 1\}$ is a unit sphere of the space R^r. We define a δ-mesh of the sphere S for a certain specified $\delta > 0$ as $\Xi = \{s_0, s_1, \ldots, s_p\}$.
We now introduce the class $\tilde{U}_H$ of all possible controls $\tilde{u}(\cdot) \in \check{U}_H$ which are such that they satisfy the relation

$$\tilde{u}(t) = \sqrt{y_{j_i}}\, s_{l_i}, \quad t \in \hat{I}_i, \quad y_{j_i} \in \Gamma^*, \quad s_{l_i} \in \Xi \tag{5.1}$$

in each interval $\hat{I}_i$ of the subdivision Γ.
The inequality

$$\Delta\sum_{i=0}^{N-1} y_{j_i} \leq \mu_0^2, \quad y_{j_i} \in [0, H^2] \tag{5.2}$$

is then true.
The attainability set of system (1.1), which is determined by the class of controls $\tilde{U}_H$, is denoted by $\tilde{X}_H(t; t_0, x_0)$.

Assertion 5. The following inequality holds

$$\alpha(\check{X}_H(t;t_0,x_0),\ \tilde{X}_H(t;t_0,x_0))\leq c_0\gamma_2(\theta-t_0)H\delta,\quad t\in I \tag{5.3}$$

The constant $c_0 > 0$ is defined by relation (2.2).

Proof. By virtue of the definition of the δ-mesh, the control $\tilde{u}(\cdot)\in\tilde{U}_H$, which satisfies the inequality

$$\left\|\check{u}(t)-\tilde{u}(t)\right\|\leq\delta\sqrt{y_{j_i}},\quad t\in\hat{I}_i \tag{5.4}$$

is found for any control $\check{u}(\cdot)\in\check{U}_H$.

Suppose $\check{x}(t)$ and $\tilde{x}(t)$ are the motions of system (1.1) which generate the controls $\check{u}(t)$ and $\tilde{u}(t)$ respectively. Then, the following inequality is satisfied

$$\check{x}(t)-\tilde{x}(t)=\int_{t_0}^{t}(f(\tau,\check{x}(\tau))-f(\tau,\tilde{x}(\tau)))d\tau+\int_{t_0}^{t}(B(\tau,\check{x}(\tau))-$$

$$-B(\tau,\tilde{x}(\tau)))\check{u}(\tau)d\tau+\int_{t_0}^{t}B(\tau,\tilde{x}(\tau))(\check{u}(\tau)-\tilde{u}(\tau))d\tau,\quad t\in I$$

By virtue of condition **A**, we therefore obtain

$$\varepsilon(t)\leq h(t)+\int_{t_0}^{t}\psi(\tau)\varepsilon(\tau)d\tau,\quad t\in I$$
$$\varepsilon(t)=\left\|\check{x}(t)-\tilde{x}(t)\right\|,\quad \psi(t)=L_1+L_2\left\|\check{u}(\tau)\right\|,\quad h(t)=\int_{t_0}^{t}\left\|B(\tau,\tilde{x}(\tau))\right\|\cdot\left\|\check{u}(\tau)-\tilde{u}(\tau)\right\|d\tau \tag{5.5}$$

By virtue of condition **B** and inequality (5.4), the estimate

$$h(t)\leq\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\left\|B(\tau,\tilde{x}(\tau))\right\|\cdot\left\|\check{u}(\tau)-\tilde{u}(\tau)\right\|d\tau\leq\sum_{i=0}^{k}\gamma_2\Delta\sqrt{y_{j_i}}\,\delta\leq\gamma_2(\theta-t_0)H\delta,\quad t\in\hat{I}_i$$

holds for the function $h(t)$.

Then, on applying Gronwall's lemma to inequality (5.5), we obtain the estimate

$$\varepsilon(t)\leq c_0\gamma_2(\theta-t_0)H\delta,\quad t\in I \tag{5.6}$$

The constant $c_0 > 0$ is defined by relation (2.2).

Since $\tilde{U}_H\subset\check{U}_H$, the validity of the assertion follows from relation (5.6).

6. We will now discuss the problem of the approximate calculation of the attainability set $\tilde{X}_H(\theta; t_0, x_0)$. All the argument and the estimates which will be obtained hold for the general case $X_H(t; t_0, x_0)$, $t\in I$.

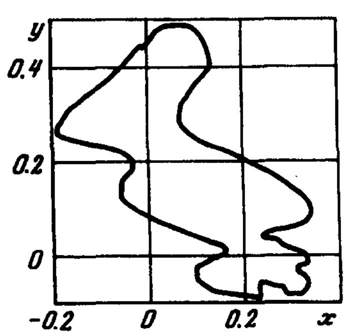

Fig. 1.

We shall make the Euler broken line

$$z(t)=z(t_i)+(t-t_i)(f(t_i,z(t_i))+B(t_i,z(t_i))\tilde{u}(t_i)),\quad z(t_0)=\tilde{x}(t_0)=x_0,\quad t\in\hat{I}_i \tag{6.1}$$

correspond to any motion $\tilde{x}(\cdot)$ of system (1.1) which is generated by a control $\tilde{u}(\cdot)\in\tilde{U}_H$.

The set of values of $z(\theta)$ of the Euler broken lines $z(\cdot)$ which are generated by all possible controls $\tilde{u}(\cdot)\in\tilde{U}_H$ is denoted by $Z(\theta;t_0,x_0)$.

Note that the values of $z(\theta)$ of the Euler broken lines (6.1) can be calculated using the recurrence formula

$$\begin{gathered}z(t_{i+1})=z(t_i)+\Delta[f(t_i,z(t_i))+B(t_i,z(t_i))\sqrt{y_{j_i}}\,s_{l_i}]\\ z(t_0)=x_0,\quad y_{j_i}\in\Gamma^*,\quad s_{l_i}\in\Xi\end{gathered} \tag{6.2}$$

We now introduce the notation

$$K^*(\Delta)=\max_{|t^*-t_*|\leqslant\Delta,\ \|x^*-x_*\|\leqslant\Delta}\|f(t^*,x^*)-f(t_*,x_*)\| \tag{6.3}$$

$$\xi^*(\Delta)=K^*(\varphi(\Delta))+H\omega^*(\varphi(\Delta)),\quad \xi(\Delta)=\Delta\xi^*(\Delta) \tag{6.4}$$

$$\begin{gathered}L=L_1+HL_2,\quad \hat{c}=(\theta-t_0)\exp[L(\theta-t_0)]\\ (t^*,x^*)\in D,\ (t_*,x_*)\in D,\quad \Delta\geqslant 0\end{gathered} \tag{6.5}$$

The functions $\omega^*(\Delta)$, $\varphi(\Delta)$ are defined by relations (3.2) and (3.3), respectively.

Assertion 6. The following inequality holds

$$\alpha(\tilde{X}_H(\theta;t_0,x_0),\ Z(\theta;t_0,x_0))\leqslant\hat{c}\xi^*(\Delta) \tag{6.6}$$

Proof. The inequality

$$\varepsilon_1\leqslant\int_{t_0}^{t_1}\|f(\tau,\tilde{x}(\tau))-f(t_0,\tilde{x}(t_0))\|d\tau+\int_{t_0}^{t_1}\|(B(\tau,\tilde{x}(\tau))-B(t_0,\tilde{x}(t_0)))\tilde{u}(t_0)\|d\tau$$

holds for $\varepsilon_1=\|\tilde{x}(t_1)-z(t_1)\|$. Hence, taking account of the fact that $\tilde{u}(t_0)=\sqrt{(y_{j_0})}\in\Gamma^*, s_{l_0}\in\Xi$ and using arguments which are analogous to those used in obtaining inequalities (3.8) and (3.9), it can be shown that $\varepsilon_1\leqslant\Delta\xi^*(\Delta)$.

For $\varepsilon_2=\|\tilde{x}(t_2)-z(t_2)\|$, the following estimate is true

$$\begin{aligned}\varepsilon_2&\leqslant\|\tilde{x}(t_1)-z(t_1)\|+\int_{t_1}^{t_2}\|f(\tau,\tilde{x}(\tau))-f(t_1,\tilde{x}(t_1))\|d\tau+\\&+\int_{t_1}^{t_2}\|(B(\tau,\tilde{x}(\tau))-B(t_1,\tilde{x}(t_1)))\sqrt{y_{j_1}}\,s_{l_1}\|d\tau+\int_{t_1}^{t_2}\|f(t_1,\tilde{x}(t_1))-f(t_1,z(t_1))\|d\tau+\\&+\int_{t_1}^{t_2}\|(B(t_1,\tilde{x}(t_1))-B(t_1,z(t_1)))\sqrt{y_{j_1}}\,s_{l_1}\|d\tau\leqslant\xi(\Delta)+\xi(\Delta)+\Delta L_1\xi(\Delta)+\Delta L_2H\xi(\Delta)\leqslant\\&\leqslant\xi(\Delta)\exp[L\Delta]+\xi(\Delta)\end{aligned}$$

Similarly, the estimate

$$\varepsilon_3\leqslant\xi(\Delta)\exp[L(\Delta+\Delta)]+\xi(\Delta))\exp[L\Delta]+\xi(\Delta)$$

holds for $\varepsilon_3=\|\tilde{x}(t_3)-z(t_3)\|$.

Finally for $\varepsilon_N=\|\tilde{x}(t_N)-z(t_N)\|=\|\tilde{x}(\theta)-z(\theta)\|$, we obtain

$$\varepsilon_N \leq \exp[L(\theta - t_0)] \sum_{i=0}^{N-1} \xi(\Delta) = \hat{c}\xi^*(\Delta) \tag{6.7}$$

Since the class of controls $\tilde{U}_H$ consists of piecewise-constant functions of the form (5.1), the validity of the assertion follows from relation (6.7).

7. The validity of the following theorem follows from Assertions 2–6.

Theorem. The following estimate holds

$$\begin{aligned}\alpha(X(\theta;t_0,x_0),\ Z(\theta;t_0,x_0)) \leq 2c_0\gamma_2\mu_0^2 / H + c_0\varkappa(\Delta) + \\ + c_0\gamma_2(\theta - t_0)\sqrt{\Delta^*} + c_0\gamma_2(\theta - t_0)H\delta + \hat{c}\xi^*(\Delta)\end{aligned} \tag{7.1}$$

The constants $c_0 > 0$, $\varkappa(\Delta) > 0$, $\xi^*(\Delta) > 0$ and $\hat{c} > 0$ are defined by relations (2.2), (3.4), (6.4) and (6.5), respectively.

Corollary. For any $\varepsilon > 0$, which may be as small as desired, it is possible to choose the numbers $H > 0$, $\Delta > 0$, $\Delta^* > 0$, $\delta > 0$ such that the following inequality is satisfied

$$\alpha(X(\theta;t_0,x_0),\ Z(\theta;t_0;x_0)) \leq \varepsilon$$

Note that, if system (1.1) is autonomous, that is, it has the form

$$\dot{x}(t) = f(x(t)) + B(x(t))u$$

then $\omega^*(\Delta) = L_2\Delta$, $K^*(\Delta) = L_1\Delta$, $\xi^*(\Delta) = L\Delta$ and the estimate (7.1) takes the form

$$\begin{aligned}\alpha(X(\theta;t_0,x_0),\ Z(\theta;t_0,x_0)) \leq 2c_0\gamma_2\mu_0^2 / H + c_0[2L_2\varphi(\Delta)(\theta - t_0)^{1/2}\mu_0 + 2\gamma_2\mu_0\Delta^{1/2}] + \\ + c_0\gamma_2(\theta - t_0)\sqrt{\Delta^*} + c_0\gamma_2(\theta - t_0)H\delta - \hat{c}L\varphi(\Delta)\end{aligned}$$

The constants $\varphi(\Delta)$, $L > 0$ are defined by relations (3.2) and (6.5), respectively.

Example. Suppose the behaviour of a control system is described by the differential equation

$$\begin{aligned}\dot{x} = \frac{1}{2}\cos 100\sqrt{|y|} + 0{,}1u_1, \quad x(0) = 0 \\ \dot{y} = \frac{1}{2}\cos 100x + 0{,}1u_2, \quad y(0) = 0\end{aligned} \tag{7.2}$$

where $t \in [0; 0.07]$ and the controlling action $u = (u_1, u_2) \in R^2$ is limited by the integral constraint

$$\int_0^{0.07} \|u(t)\|^2 dt = \int_0^{0.07} [u_1(t)^2 + u_2(t)^2] dt \leq \mu_0^2 = \frac{1}{4}$$

The attainability set of control system (7.2) at the instant of time $\theta = 0.07$ is shown in Fig. 1.

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